ELSEVIER Journal of Pure and Applied Algebra 127 (1998) 147–151

JOURNAL OF PURE AND APPLIED ALGEBRA

Repleteness and the associated sheaf

Anna Bucalo, Giuseppe Rosolini*

Dipartimento di Matematica, via Dodecaneso 35, 16146 Genova, Italy

Communicated by P.T. Johnstone; received 24 April 1996; revised 23 July 1996

Abstract

We give a new categorical definition of the associated sheaf functor for a Lawvere–Tierney topology in a topos. Although the existence of such a functor is well known, the construction presented here does not resemble any other in the literature and it seems simple enough to deserve mention. We give direct comparisons with other presentations. © 1998 Elsevier Science B.V. All rights reserved.

1991 Math. Subj. Class.: 18A40, 03G30, 68Q55

Fix an object S in a cartesian closed category C, and consider the following definition of S-replete object, see [8, 9, 16]: an object Z is *S-replete* if for every $f : A \to B$ in C such that $S^f : S^B \to S^A$ is iso, one has that for every $\alpha : A \to Z$ there is a unique $\beta : B \to Z$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\alpha"'] & B \arrow[d, dotted, "\beta"] \\ & Z. \end{tikzcd}$$

One would read it as saying that Z has the unique extension property with respect to all those maps which S "believes" are isomorphisms. It is then clear that if C is a topos, and one picks S as the object Ω_j of j-closed truth values for $j : \Omega \to \Omega$ a topology on C, then the S-replete objects are exactly the j-sheaves.

In the following, we apply some known results about repleteness to produce yet another presentation for the associated sheaf functor, see [2, 3, 5–7, 10, 11, 13, 14,

* Corresponding author. E-mail: rosolini@disi.unige.it. Supported by CEE Project No. 6811, CLICS II, and MURST 40%.

PII S0022-4049(96)00151-X

17–20]. It is certainly a well-known construction, but the one we present has a crucial difference from all the others: the sheaf associated to X is carved out of its *double dual* $\Omega_j^{\Omega_j^X}$, where X appears in the covariant position. This gives the possibility to prove the reflection preserves monos (hence equalizers) without invoking injectivity in a different fashion from [10].

In the next section, we define the reflection and prove the preservation of monos. Then we recall the general steps which give left exactness of the reflector, and draw some comparisons with the work of Freyd and others.

1. The associated sheaf

Suppose C is an elementary topos and $S = \Omega_j \underset{i}{\overset{p}{\rightleftarrows}} \Omega$ is the retract of j-closed truth values for a topology j on Ω. Consider the adjunction $(\Omega_j^{(-)})^{\mathrm{op}} \dashv \Omega_j^{(-)}$. Following the common custom of writing PX for the power object Ω^X, we shall also write P_jX for Ω_j^X as this represents j-closed subobjects of X in C. Let $h_X : X \to P_j^2X = P_j(P_j(X))$ be the unit obtained by transposing the twisted evaluation. First note that

Lemma 1. *X is a j-sheaf if and only if $h_X : X \to P_j^2X$ is a j-closed mono.*

Proof. If h_X is closed, then X is a sheaf as a closed subobject of a sheaf. Conversely, just note that if Z is a sheaf, then the singleton map factors with $s : Z \rightarrowtail P_jZ$ which is closed as mono between sheaves. Then h_Z is (closed) monic as first factor of a closed mono in the diagram:

$$\begin{tikzcd} Z \arrow[r, tail, "s"] \arrow[d, "h_Z"'] & P_jZ \arrow[d, tail, shift right] \\ P_j^2Z \arrow[r, "P_j^2 s"'] & P_j^3Z \arrow[u, shift right] \end{tikzcd}$$ □

The definition of the reflection is now forced on us: for X in C take aX to be the j-closure of the image of $h_X : X \to P_j^2X$. Let $\eta_X : X \to aX$ be the factor of h_X. It is immediately seen that the assignment a is a functor and η is a natural transformation. Moreover,

Theorem 2. *For every j-sheaf Z, and every $f : X \to Z$ there is a unique $g : aX \to Z$ such that*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & aX \arrow[d, dotted, "g"] \\ & Z. \end{tikzcd}$$

Hence, a is the reflection of C into the full subcategory of j-sheaves. Moreover, a preserves monos.

Proof. Existence of g follows from the fact that η is natural and Lemma 1. Uniqueness follows from the fact that η_X has a j-dense image. As for the final part, note that if $m : X \rightarrowtail Y$ in C, then $P_j^2 m : P_j^2 X \rightarrowtail P_j^2 Y$ since $P_j m : P_j Y \to P_j X$ is epic: in fact split, as appears in

$$\begin{tikzcd}
P_j Y \arrow[r, "P_j m"] & P_j X \arrow[d, shift right, tail, "i^X"'] \\
PY \arrow[u, "p^Y"] \arrow[r, shift right, "Pm"'] & PX. \arrow[l, shift right, "\exists_m"'] \arrow[u, shift right, "p^X"']
\end{tikzcd}$$

Therefore, $a(m) : aX \to aY$ is monic as seen in the commutative diagram:

$$\begin{tikzcd}
aX \arrow[r, tail, "k_X"] \arrow[d, "a(m)"'] & P_j^2 X \arrow[d, tail, "P_j^2 m"] \\
aY \arrow[r, tail, "k_Y"] & P_j^2 Y.
\end{tikzcd}$$ □

2. Comparisons

The simplest argument to show that a is left-exact is in [7] and goes as follows:

(1) a preserves products as $\mathrm{sh}_j(C) \subset C$ is an exponential ideal.

(2) a preserves equalizers because it preserves monos; given a pullback of monomorphisms, take its pushout. The connecting map given by universality is monic. Hence, the reflector takes this to a pushout of monos (which is also a pullback) prolonged with another mono to produce a pullback.

Note that the proof in [7] makes no reference to the actual description of the associated sheaf functor. Since preservation of monos is ensured also by the fact that the inclusion $\mathrm{sh}_j(C) \subset C$ preserves injective objects; indeed, these are retracts of powers of Ω_j, hence of powers of Ω.

It is possible to prove directly that pullbacks of monos are preserved as follows: applying $\Omega_j^{(-)}$ to such a pullback gives an absolute pushout of split epis. The monadicity result obtained by Paré in [15] yields the desired pullback.

This leads to [12], where the idempotent monad induced by the adjunction $(\Omega_j^{(-)})^{\mathrm{op}} \dashv \Omega_j^{(-)}$ is considered, see also [4].

Given a monad (T, η, μ) on a category C with equalizers, the equalizer

$$QX \xrightarrow{k_X} TX \underset{T(\eta_X)}{\overset{\eta_{TX}}{\rightrightarrows}} T^2 X$$

is the underlying functor of another monad on C. The following is a remark attributed to Bkouche in [4]: the monad induced on Q is idempotent if and only if $Tk : TQ \to T^2$ is monic. The proof is by diagram chasing. That is also equivalent to the fact that the category of Q-algebras is equivalent to the full subcategory of C on those objects A which appear in an equalizer diagram of the form

$$A \longrightarrow TX \underset{g}{\overset{f}{\rightrightarrows}} TY,$$

see also [1, 3].

Thus, in the case $T = P_j^2$ one obtains a description of the associated sheaf as the equalizer of the fork

$$P_j^2 X \underset{P_j^2 \eta_X}{\overset{\eta_{P_j^2 X}}{\rightrightarrows}} P_j^4 X$$

as $\Omega_j^{(-)}$ takes monomorphisms to retractions.

Acknowledgements

We would like to thank Aurelio Carboni, Peter Freyd and Peter Johnstone for useful comments, Paul Taylor for providing the TEX-macros to typeset the diagrams.

References

[1] H. Applegate and M. Tierney, Categories with models, in: B. Eckmann, Ed., Seminar on Triples and Categorical Homology Theory, Lecture Notes in Mathematics, Vol. 80 (ETH Zürich, 1966/67) (Springer, Berlin, 1969) 156–244.
[2] M. Barr and C. Wells, Toposes, Triples and Theories (Springer, New York, 1984).
[3] F. Borceux, Handbook of Categorical Algebra (Cambridge Univ. Press, Cambridge, 1994).
[4] S. Fakir, Monade idempotente associée à une monade, C.R. Acad. Sci. Paris 270(A) (1970) 99–101.
[5] M.F. Farina, Una costruzione del fascio associato, Rend. Sci. Ist. Lomb. 110(A) (1976) 332–341.
[6] M.F. Farina and G.C. Meloni, The associated sheaf functor, Rend. Sci. Ist. Lomb. 114(A) (1980) 50–55.
[7] P. Freyd, Aspects of topoi, Bull. Austral. Math. Soc. 7 (1972) 1–76 and 467–480.
[8] J.M.E. Hyland, First steps in synthetic domain theory, in: A. Carboni, M. Pedicchio and G. Rosolini, Eds., Category Theory, Lecture Notes in Mathematics, Vol. 1488 (Como, 1990) (Springer, Berlin, 1992) 131–156.
[9] J.M.E. Hyland and E. Moggi, The S-replete construction, in: D. Pitt, D.E. Rydeheard and P. Johnstone, Eds., Category Theory and Computer Science, Lecture Notes in Computer Science, Vol. 953 (Cambridge, 1995) (Springer, Berlin, 1995) 96–116.
[10] P. Johnstone, Topos Theory, L.M.S. Monographs, Vol. 10 (Academic Press, London, 1977).
[11] A. Kock and G.C. Wraith, Elementary Toposes, Lecture Notes Series, Vol. 30 (Århus Universitet, Århus, 1971).
[12] J. Lambek and B.A. Rattray, Localization and sheaf reflectors, Trans. Amer. Math. Soc. 210 (1975) 279–293.
[13] F.W. Lawvere, Introduction, in: F.W. Lawvere, C. Maurer and G.C. Wraith, Eds., Toposes, Algebraic Geometry and Logic, Lecture Notes in Mathematics, Vol. 274 (Springer, Berlin, 1972) 1–12.
[14] S. Mac Lane and I. Moerdijk, Sheaves in Geometry and Logic (Springer, New York, 1992).

[15] R. Paré, Colimits in topoi, Bull. Amer. Math. Soc. 80 (1974) 556–561.
[16] P. Taylor, The fixed point property in synthetic domain theory, in: G. Kahn, Ed., Proc. 6th Symposium on Logic in Computer Science, Amsterdam (1991) 152–160.
[17] M. Tierney, Axiomatic sheaf theory: some constructions and applications, in: Categories and Commutative Algebra (CIME III ciclo, 1971) (Edizioni Cremonese, Roma, 1973) 249–326.
[18] B. Veit, A proof of the associated sheaf theorem by means of categorical logic, J. Symbolic Logic 46 (1981) 45–55.
[19] F.-J. de Vries, Type theoretical topics in topos theory, Ph.d. Thesis, Rijks Universiteit, Utrecht, 1989.
[20] G.C. Wraith, Lectures on elementary topoi, in: F.W. Lawvere, C. Maurer and G.C. Wraith, Eds., Model Theory and Topoi, Lecture Notes in Mathematics, Vol. 445 (Springer, Berlin, 1975) 114–206.